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Fibonacci Functional Sequence Generating Functions

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Abstract:
The purpose of this paper is to generalize some results of Horadam and to extend some identities due to Carlitz in the context of functional difference equations and their associated generating functions.

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AMS Classifications: 11B39

1. Introduction

In a previous paper [1] a functional difference equation was defined in order to generalize aspects of Horadam's generalized sequence of numbers, $\{w_n(a,b;p,q)\}$, defined by the second order linear homogeneous recurrence relation

$$w_n = pw_{n-1} - qw_{n-2},\ n > 2. \tag{1.1}$$

with initial conditions $w_0 = a, w_1 = b$ [2] with fundamental properties [3,4]. The element of the functional difference equation in question is defined as

$$w_n(x) = \sum_{k=0}^{\infty} w_{n+k} \frac{x^k}{k!},\ (n = 0,1,2,...), \tag{1.2}$$

so that we get the differential equation

$$\frac{d}{dx} w_n(x) = w_{n+1}(x), \tag{1.3}$$

and the difference equation

$$w_n(x) = pw_{n-1}(x) - qw_{n-2}(x),\ n > 2. \tag{1.4}$$

The purpose of this note is to extend this to generating functions which is done in Section 3 after some preliminary notational clarifications in Section 2 where the functional sequence generating function is defined by

$$w_n^*(x) = w_n^*(x,\lambda) = \sum_{k=0}^{\infty} w_{n+k}\binom{x}{k}\lambda^k \tag{1.5}$$

and

$$w_k^*(x,y) = \sum_{n=0}^{\infty}\left(w_n^*(x)\right)^k y^n = \sum_{n=0}^{\infty} w_n^{*k}(x)y^n, \tag{1.6}$$

which is slightly more general than that studied by Carlitz [5], which is itself a generalization of the results of Elmore [6]. The general term of $\{w_n(a,b;p,q)\}$ is well known:

$$w_n = A\alpha^n + B\beta^n \tag{1.7}$$

where α, β are the zeroes of $x^2 - px + q$ and $p = \alpha + \beta, q = \alpha\beta, d = \alpha - \beta, E = pab - qa^2 - b^2$ and from the initial conditions

$$A = \frac{b - a\beta}{d}, B = \frac{a\alpha - \beta}{d},$$

so that

$$E = ABd^2. \tag{1.8}$$

The result in this note then follows from Carlitz' method for finding the k^{th} power generating function of a second order recursive sequence [7], which was generalized by the present writer [8]. In order to investigate factorizations, Brillhart *et al* [9] built on the well-known identity connecting the Fibonacci and Lucas, namely,

$$F_{2n} = F_n L_n;$$

Shannon *et al* [10] considered a type of functional analogue of this in the form

$$f\left(2 - x^2\right) = f(x)f(-x).$$

Other aspects of Fibonacci functional equations, particularly stability issues, may be found in Lather and Singh [11], Parizi and Gordji [12], Hyers and Rassias [13].

2. Notational Preliminaries

Now from (1.2) and (1.5) we further define

$$w_n(x) = \sum_{k=0}^{n}\left(A\alpha^{n+k} + B\beta^{n+k}\right)\frac{x^k}{k!}$$
$$= A\alpha^n e^{\alpha x} + B\beta^n e^{bx}.$$

It follows immediately that

$$\sum_{k=0}^{\infty} w_{n+k}(x)\frac{y^k}{k!} = \sum_{k=0}^{\infty}\left(A\alpha^{n+k}e^{\alpha x} + B\beta^{n+k}e^{\beta x}\right)\frac{y^k}{k!}$$
$$= A\alpha^n e^{\alpha(x+y)} + B\beta^n e^{\beta(x+y)}$$
$$= w_n(x+y).$$

Methods of solving functional difference equations may be found in Shannon *et al* [14]. We can also define

$$w_n^*(x) = w_n^*(x,\lambda) = \sum_{k=0}^{\infty} w_{n+k}\binom{x}{k}\lambda^k \tag{2.1}$$

As previously stated in (1.5). Then

$$w_n^*(0) = w_n,$$

and

$$\begin{aligned} w_{n+1}^* &= \sum_{k=0}^{\infty} w_{n+k+1}\binom{x}{k}\lambda^k \\ &= \sum_{k=0}^{\infty}\left(pw_{n+k} - qw_{n+k-1}\right)\binom{x}{k}\lambda^k \\ &= pw_n^*(x) - qw_{n-1}^*(x) \end{aligned} \tag{2.2}$$

as expected. Moreover, from (2.1) we can define a functional operator:

$$\begin{aligned} \Delta_x w_n^*(x) &= w_n^*(x+1) - w_n^*(x) \\ &= \sum_{k=0}^{\infty} w_{n+k}\left\{\binom{x+1}{k} - \binom{x}{k}\right\}\lambda^k \\ &= \lambda\sum_{k=1}^{\infty} w_{n+k}\binom{x}{k-1}\lambda^{k-1} \\ &= \lambda\sum_{k=0}^{\infty} w_{n+k+1}\binom{x}{k}\lambda^k \end{aligned}$$

so that

$$\Delta_x w_n^*(x,\lambda) = \lambda w_{n+1}^*(x,\lambda) \tag{2.3}$$

which shows that the power series in the definition (2.1) converges for sufficiently small λ. We now formally define

$$w_k^*(x,y) = \sum_{n=0}^{\infty}\left(w_n^*(x)\right)^k y^n = \sum_{n=0}^{\infty} w_n^{*k}(x) y^n, \tag{2.4}$$

as in (1.6), so that

$$w_k^*(0,y) = w_k(y),$$

the properties of which have been studied by Horadam [15]. We shall also utilize a_{kj} defined by Carlitz [7] as

$$\begin{aligned}\sum_{k=2j}^{\infty} a_{k,j} x^{k-2j} &= \left(\frac{1}{1-px+qx^2}\right)^j \\ &= \left(\frac{w_0(x)}{a+(b+pa)x}\right)^j.\end{aligned} \tag{2.5}$$

3. Functional Sequence Generating Function

Following another idea of Carlitz, we also consider $W^*(x,z)$ defined as

$$W^*(x,z) = \sum_{k=1}^{\infty}\left(1-\alpha^k y\right)\left(1-\beta^k y\right) w_k^*(x,y)\frac{z^k}{k}. \tag{3.1}$$

Then

$$\begin{aligned}W^*(x,z) &= \sum_{k=1}^{\infty}\left(1-\alpha^k y\right)\left(1-\beta^k y\right) w_k^*(x,y)\frac{z^k}{k} \\ &= \sum_{k=1}^{\infty}\left(1-\alpha^k y\right)\left(1-\beta^k y\right)\frac{z^k}{k}\sum_{j=0}^{\infty}\left(w_j^*(x)\right)^k y^j \\ &= \sum_{k=1}^{\infty}\left(1-\left(\alpha^k+\beta^k\right)y+q^k y^2\right)\frac{z^k}{k}\sum_{j=0}^{\infty}\left(w_j^*(x)\right)^k y^j \\ &= -\sum_{j=0}^{\infty} y^j \log\left(1-w_j^*(x)z\right)+\sum_{j=0}^{\infty} y^{j+1}\log\left(1-\alpha w_j^*(x)z\right)+\sum_{j=0}^{\infty} y^{j+1}\log\left(1-\beta w_j^*(x)z\right) \\ &\quad -\sum_{j=0}^{\infty} y^{j+2}\log\left(1-q w_j^*(x)z\right) \\ &= -\log\left(1-w_0^*(x)z\right)+y\log\left(1-q w_{-1}^*(x)z\right)-y\sum_{j=0}^{\infty} y^j \log\left(1-w_{j+1}^*(x)z\right) \\ &\quad -y\sum_{j=0}^{\infty} y^j\log\left(1-q w_{j-1}^*(x)z\right)+\sum_{j=0}^{\infty} y^{j+1}\log\left(1-p w_j^*(x)z+q w_j^{*2}(x)z^2\right)\end{aligned}$$

From equation (4.1) in [1], we have a generalization of Simson's identity:

$$w_{j+1}^*(x)w_{j-1}^*(x) = w_j^{*2}(x) + q^{j-1}EE_w^x \tag{3.2}$$

in which

$$E_w = 1 + p\lambda + q\lambda^2, \tag{3.3}$$

and 'E' (in 1.6) and 'E_w' (in 3.3) turn out to be fortuitous choices of symbols because in the subsequent results they have a role somewhat analogous to the exponential function. Leonard Carlitz and Alwyn Horadam were both well known for the serendipity of their development of appropriate notation to suggest elegant analogies.

It follows that

$$\left(1 - w_{j+1}^*(x)\right)z\left)(1-\right) = 1 - pw_j^*(x)z + \left(qw_j^{*2}(x) + q^j EE_w^x\right)z^2, \tag{3.4}$$

because

$$\left(1 - w_{j+1}^*(x)\right)z\left)\left(1 - qw_{j-1}^*(x)z\right) = 1 - \left(w_{j+1}^*(x)\right)z + qw_{j-1}^*(x)z\right) + q\left(w_{j+1}^*(x)w_{j-1}^*(x)\right)z^2$$
$$= 1 - \left(pw_j^*(x)z\right) + \left(qw_j^{*2}(x) + q^j EE_w^x\right)z^2.$$

Thus

$$W^*(x,z) = -\log\left(1 - w_0^*(x)z\right) + y\log\left(1 - w_{-1}^*(x)z\right) - y\sum_{j=0}^{\infty} y^j \log\left(1 + \frac{q^j EE_w^x z^2}{1 - pw_j^*(x)z + qw_j^{*2}(x)z^2}\right)$$
$$= -\log\left(1 - w_0^*(x)z\right) + y\log\left(1 - w_{-1}^*(x)z\right) - y\sum_{j=0}^{\infty} y^j \sum_{s=1}^{\infty} \frac{(-1)^s}{s}\left(\frac{q^j EE_w^x z^2}{1 - pw_j^*(x)z + qw_j^{*2}(x)z^2}\right)$$
$$= -\log\left(1 - w_0^*(x)z\right) + y\log\left(1 - w_{-1}^*(x)z\right) - y\sum_{j=0}^{\infty} y^j \sum_{s=1}^{\infty} \frac{(-1)^s q^{js} E^s E_w^{xs}}{s} z^{2s} \sum_{k=2s}^{\infty} a_{ks}\left(w_j^*(x)z\right)^{k-2s}$$
$$= -\log\left(1 - w_0^*(x)z\right) + y\log\left(1 - w_{-1}^*(x)z\right) - y\sum_{j=0}^{\infty} y^j \sum_{k=1}^{\infty} z^k \sum_{s=1}^{\lfloor \frac{k}{2} \rfloor} \frac{(-E)^s q^{js} E_w^{xs}}{s} \sum_{k=2s}^{\infty} a_{ks}\left(w_j^*(x)z\right)^{k-2s}$$
$$= -\log\left(1 - w_0^*(x)z\right) + y\log\left(1 - w_{-1}^*(x)z\right) - y\sum_{k=1}^{\infty} z^k \sum_{s=1}^{\lfloor \frac{k}{2} \rfloor} \frac{(-E)^s q^{js} E_w^{xs} a_{ks} W_{k-2s}^*\left(x, q^s y\right)}{s}.$$

Comparing coefficients of z^k we get

$$\left(1 - v_k y + q^k y^2\right)W_k^*(x,y) = w_0^{*k}(x) - yq^k w_{-1}^{*\,k}(x) + ky\sum_{s=1}^{\lfloor \frac{k}{2} \rfloor} \frac{(-E)^s E_w^{x^s} a_{ks} W_{k-2s}^*(x, q^s y)}{s}, \tag{3.5}$$

where

$$v_k = \alpha^k + \beta^k$$

is a Lucas primordial number. When $x = 0$, (3.4) reduces to

$$\left(1-v_k y+q^k y^2\right)w_k(y)=a^k-y^k(pa-b)^k+ky\sum_{s=1}^{\lfloor\frac{k}{2}\rfloor}\frac{(-E)^s a_{ks}w_{k-2s}(q^s y)}{s},$$

which agrees with equation (33) of Horadam [2], since E_w disappears when x is zero. It should also be possible to develop an extension of a Pythagorean theorem for $w_n^*(x)$ [16].

4. Concluding Comments

Carlitz [17], Larcombe *et al* [18] and Atanassov *et al* [19] have also considered the arithmetic properties of other sequences. Hung *et al* [20] and Mangel [21;22] have applications of functional difference equations in medicine and biology. A bonus is that it is only a small step from the results in [1] to some interesting student projects [23]. We conclude, however, with a generalization of the functional difference equation which suggests further extensions along the lines of the preceding sections. From equations (2.3) and (2.4):

$$\begin{aligned} w_n^*(x) &= \sum_{k=0}^{\infty}\left\{A\alpha^n\binom{x}{k}(\alpha\lambda)^k+B\beta^n\binom{x}{k}(\beta\lambda)^k\right\} \\ &= A\alpha^n(1+\lambda\alpha)^x+B\beta^n(1+\lambda\beta)^x. \end{aligned} \tag{4.1}$$

Similarly then,

$$\begin{aligned} w_n^*(x+y) &= A\alpha^n(1+\lambda\alpha)^{x+y}+B\beta^n(1+\lambda\beta)^{x+y} \\ &= \sum_{k=0}^{\infty}\left\{A\alpha^{n+k}\binom{x}{k}(\alpha\lambda)^k+B\beta^{n+k}\binom{x}{k}(\beta\lambda)^k\right\} \\ &= \sum_{k=0}^{\infty}w_{n+k}^*(x)\binom{y}{k}\lambda^k, \end{aligned} \tag{4.2}$$

the last factor of which is missing from the comparable equation (3.3) of Carlitz [24]. Furthermore, if we generalize the results from the second order cases to arbitrary order r, we obtain

$$w_n^*(x)=w_n^*(x,\lambda)=\sum_{k=0}^{\infty}w_{n+k}^{(r)}\binom{x}{k}\lambda^k$$

where

$$w_n^*(0)=w_n^{(r)}=\sum_{k=0}^{\infty}(-1)^k P_{r,n-k+1}w_k^{(r)},$$

in which the $P_{r,k}$ are arbitrary integers, then the arbitrary order extensions of (4.2) can be given by

$$w_n^*(x+y) = \sum_{k=o}^{\infty} w_{n+k}^*(x)\binom{y}{k}\lambda^k$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} w_{n+j+k}^{(r)}\binom{x}{j}\binom{y}{k}\lambda^{j+k}$$

and

$$w_n^*(x+y+z) = \sum_{k=o}^{\infty} w_{n+k}^*(x+y)\binom{z}{k}\lambda^k$$
$$= \sum_{i=o}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} w_{n+i+j+k}^{(r)}\binom{x}{i}\binom{y}{j}\binom{z}{k}\lambda^{i+j+k}$$

and so on. Similarly, this work could also be extended to explore analytic extensions in the complex plane [25].

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